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Finite Elements for Maxwell's Equations

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Abstract

Maxwell's equations describe the phenomena of electromagnetism. In the first chapter of this thesis we will derive these equations. For the analysis we will need to obtain a common structure for different type of problems, which results in the curl–curl problem.

The curl–curl problem is the starting point to obtain a weak formulation. Therefore we will need an integration by parts formula, a trace operator and a function space such that the expressions in the weak formulation are well defined.
A helpful tool will be the de Rham complex, which summarizes many properties of the function spaces we will consider.
Looking forward to the finite elements, we need to find an interface condition such that a function in a big domain can be decomposed in functions living in restricted parts.

In the last chapter of this thesis we will create finite elements for Maxwell's equations. An important aim is to derive also an exact sequence for the discrete case.
Thus we will define nodal interpolation operators and consider a commuting property. This commuting property helps us constructing the functionals for each element space. Moreover, the commutation property satisfies that our discrete sequence is exact.
An important tool of the FEM is the mapping trick. Here we will consider affine transformations and prove, that they are preserving the degrees of freedom.
Finally an interpolation error estimate is formulated.

Zusammenfassung

Die Maxwell–Gleichungen beschreiben die Phänomene des Elektromagnetismus. Im ersten Kapitel der Arbeit leiten wir diese her. Um die Gleichungen zu untersuchen benötigen wir eine gemeinsame Struktur verschiedener Problemklassen – das curl–curl Problem.

Das curl–curl Problem ist der Ausgangspunkt für unsere Variationsformulierung. Daher benötigen wir eine Formel der partiellen Integration, einen Spuroperator und einen Sobolevraum, dass die Ausdrücke in unserer schwachen Formulierung wohldefiniert sind.
Ein wichtiges Hilfsmittel wird der de Rham Komplex sein. Dieser fasst die wichtigsten Eigenschafen unserer Funktionenräume kompakt zusammen.
Im Hinblick auf die Finiten Element brauchen wir noch eine Bedingung für die Funktionen an einer Grenzfläche, sodass eine Funktion in einem größeren Gebiet in Funktionen kleiner Gebiete unterteilt werden kann.

Im letzten Kapitel der Arbeit entwerfen wir Finite Elemente für die Maxwell–Gleichungen. Ein bedeutendes Ziel wird die Erschaffung einer exakten Sequenz für den endlichdimensionalen Fall sein.
Deswegen werden wir einen nodalen Interpolationsoperator definieren und betrachten eine Kommutationsbedingung. Diese hilft uns bei der Konstruktion der Funktionale für die einzelnen lokalen Funktionenräume. Darüberhinaus ist die Kommutationsbedingung der Grund das unsere Sequenz exakt ist.
Ein wichtiges Werkzeug bei der FEM ist das Abbilden auf ein Referenzelement. Hier werden wir nur affin lineare Abbildungen betrachten und überprüfen, ob sie die Anforderungen unserer Funktionenräume erfüllen.
Zum Schluss ist noch eine Fehlerabschätzung für den nodalen Interpolationsoperator angegeben.

Acknowledgments

I would like to thank my supervisor Dipl.–Ing. Dr. Martin Neumüller, spending hours of time eliminating any of my questions. There occurred lots of while writing this thesis. I wish him all the best for his future career aside the university.

Florian Kagerer
Asten, July 2018

Contents

Chapter 1

Derivation of Maxwell's Equations

In 1862, James Clerk Maxwell (*1831 - †1879) published "*A Treatise on Electricity and Magnetism*". In this paper he described the interaction between magnetic fields and electric fields.

Maxwell's most important research was working out and modelling a set of coupled partial differential equations describing electromagnetic phenomena. He used earlier research works and results of great physicists like Michael *Faraday* and André–Marie *Ampère*. Using these equations, Maxwell confirmed the suggestion from the early 19^{th} century, that there is a "reasonable" model combining electricity and magnetism.

In this chapter we want to derive Maxwell's equations and reformulate them to a common structure, the curl-curl problem, which will be our starting point for the weak formulation.

1.1 Equations of Magnetic Fields

Maxwell introduced three vector functions of position $\underline{x} \in \mathbb{R}^3$ and time $t \in \mathbb{R}$ to describe a magnetic field,

$$
\begin{array}{lll}
\mathcal{H} & \ldots \quad \text{magnetic field intensity (resp. magnetic field)} & \left[\frac{A}{m}\right], \\
\mathcal{B} & \ldots \quad \text{magnetic induction (resp. magnetic flux density)} & [T] = \left[\frac{N}{A \cdot m}\right], \\
\mathcal{J} & \ldots \quad \text{total current density} & \left[\frac{A}{m^2}\right].
\end{array}
$$

Note, the SI units denotes Ampère $[A]$, meter $[m]$, and Newton $[N]$.

In the following we will use laws of physics and laws of properties of electromagnetism to get a step closer to Maxwell's equations.

Magnetic Field is Solenoidal

The magnetic flux density $\mathcal{B}$ is illustrated by closed magnetic field lines. Hence, the magnetic field is *solenoidal*, i.e. it is source free, also $\mathcal{B}$ has no sources. This is mathematically written as

$$\int\limits_{\partial V} \mathcal{B} \cdot \underline{n} \, \mathrm{d}s_x = 0 \quad \text{for any bounded volume } V \subseteq \mathbb{R}^3,$$

where $\underline{n}$ is an outer unit normal vector. In other words, $\mathcal{B}$ is conservative through the surface of V.

Ampère's Law

Current through a wire causes a magnetic field. Ampère's law (ger.: *Durchflutungssatz*) says that the sum of the magnetic field along a closed path (∂S) is proportional to the current causing the magnetic field through the enclosed surface S, that means

$$\int\limits_{\partial S} \mathcal{H} \cdot \underline{\tau} \, \mathrm{d}\ell_x = \int\limits_{S} \mathcal{J} \cdot \underline{n} \, \mathrm{d}s_x,$$

where $\underline{\tau}$ is a unit tangential vector. This relation is illustrated in Figure 1.1.

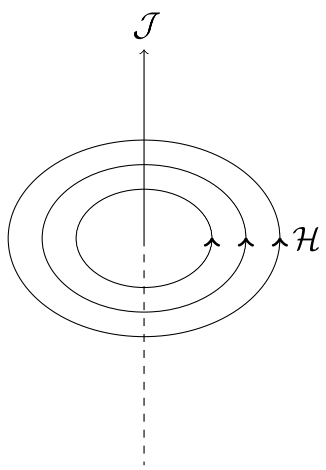


Figure 1.1: Ampère's Law

Maxwell generalized Ampère's law: since the change of the displacement field $\mathcal{D}$ leads to a flow of current we have

$$\int\limits_{\partial S} \mathcal{H} \cdot \underline{\tau} \, \mathrm{d}\ell_x = \int\limits_{S} \mathcal{J} \cdot \underline{n} \, \mathrm{d}s_x + \int\limits_{S} \frac{\partial \mathcal{D}}{\partial t} \cdot \underline{n} \, \mathrm{d}s_x.$$

Remark. This generalization made by Maxwell is the reason why the equations are called after him.

Material Law

The system is under determined, therefore we need a material law which relates the properties $\mathcal{B}$ and $\mathcal{H}$. For this reason materials are distinguished by their magnetic behaviour:

- diamagnetic materials (e.g. Copper, Silver): magnetization opposes magnetic field
- paramagnetic materials (e.g. Aluminium): magnetization in the same direction as magnetic field
- ferromagnetic materials (e.g. Iron): magnetization can be independent of magnetic field; complex relation
- superconductors: may have diamagnetic properties under certain circumstances or have a complex hysteretic dependence of $\mathcal{B}$ and $\mathcal{H}$

Here we assume a linear relation (therefore we have either diamagnetic or paramagnetic materials), so we can describe

$$\mathcal{B} = \mu \mathcal{H},$$

where μ is called *permeability*.

1.2 Equations of Electric Fields

The phenomena of electric fields are described by

$\mathcal{E}$	...	electric field intensity (resp. electric field)	$\left[\frac{V}{m}\right]$,
$\mathcal{D}$	...	electric displacement field (resp. displacement current density)	$\left[\frac{As}{m^2}\right]$,
$\boldsymbol{j}_{\mathrm{c}}$	...	electric current density	$\left[\frac{A}{m^2}\right]$,
ρ	...	charge density	$\left[\frac{As}{m^3}\right]$.

Note, the SI units denotes Volt $[V]$, meter $[m]$, and seconds $[s]$.

Faraday's Induction Law

Consider a wire which forms a closed loop ∂S. Faraday discovered that a change of the magnetic flux $\mathcal{B}$ trough the surface S, spanned by the wire, induces a voltage in the loop and creates an electric field $\mathcal{E}$ – see Figure 1.2. That means

$$\int\limits_{S} \frac{\partial \mathcal{B}}{\partial t} \cdot \underline{n} \, \mathrm{d}s_x = - \int\limits_{\partial S} \mathcal{E} \cdot \underline{\tau} \, \mathrm{d}\ell_x .$$

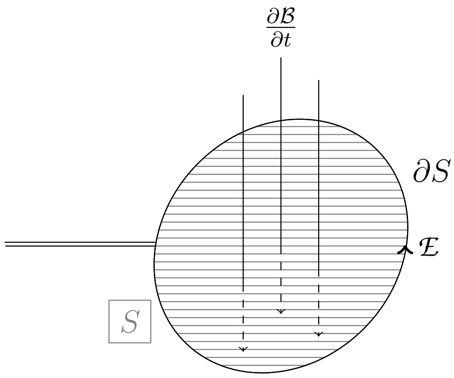


Figure 1.2: Faraday's Induction Law

Gauss's Law

This law describes how electric charges cause an electric field. It has the form

$$\int_{\partial V} \mathcal{D} \cdot \underline{n} \, \mathrm{d}s_x = \int_V \rho \, \mathrm{d}x. \tag{1.1}$$

Ohm's Law

In conducting materials, e.g. copper, the electric field induces a current with density $\boldsymbol{j}_\mathrm{c}$. Ohm's law says that $\boldsymbol{j}_\mathrm{c}$ and $\mathcal{E}$ are proportional, i.e.

$$\boldsymbol{j}_\mathrm{c} = \sigma \mathcal{E} \quad \text{with} \quad \mathcal{J} = \boldsymbol{j}_\mathrm{c} + \boldsymbol{j}_\mathrm{i},$$

where σ is called *electric conductivity* and $\boldsymbol{j}_\mathrm{i}$ the impressed current density.

Material Law

The electric field density $\mathcal{E}$ and the corresponding displacement current density $\mathcal{D}$ are coupled by the *electric permittivity* ε, i.e.

$$\mathcal{D} = \varepsilon \mathcal{E}.$$

1.3 Maxwell's Equations

In the next step we want to derive Maxwell's equations in differential formulation. They are a system of four PDE's which describe all phenomena of electromagnetism. We assume smooth fields, so we can apply Gauss's theorem and Stoke's theorem.

1.3.1 Reformulating Magnetic Field Equations

So, by *Gauss's theorem* we can reformulate the property that $\mathcal{B}$ is conservative on V to

$$0 = \int_{\partial V} \mathcal{B} \cdot \underline{n} \, \mathrm{d}s_x \overset{\text{Gauss}}{=} \int_V \operatorname{div} \mathcal{B} \, \mathrm{d}x \qquad \forall V \subseteq \mathbb{R}^3.$$

Because of the *fundamental lemma of calculus of variations*, it follows

$$\operatorname{div} \mathcal{B} = 0.$$

Applying *Stokes' theorem* on Ampère's law we obtain

$$\int_S \frac{\partial \mathcal{D}}{\partial t} \cdot \underline{n} \, \mathrm{d}s_x + \int_S \mathcal{J} \cdot \underline{n} \, \mathrm{d}s_x = \int_{\partial S} \mathcal{H} \cdot \underline{\tau} \, \mathrm{d}\ell_x = \int_S \operatorname{curl} \mathcal{H} \cdot \underline{n} \, \mathrm{d}s_x.$$

The formula above is valid for all S and the integrand is continuous. Due to the fundamental lemma of calculus of variations we obtain

$$\operatorname{curl} \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J}. \tag{1.2}$$

1.3.2 Reformulating Electric Field Equations

We use Stoke's theorem to get Faraday's law in differential form, namely

$$-\operatorname{curl} \mathcal{E} = \frac{\partial \mathcal{B}}{\partial t}.$$

Applying Gauss's theorem on (1.1) in combination with the fundamental lemma of calculus of variations, we get

$$\operatorname{div} \mathcal{D} = \rho. \tag{1.3}$$

1.3.3 Result – Maxwell's Equations

We derived Maxwell's equations, namely

$$\operatorname{curl} \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J}_{\mathrm{tot}}, \tag{1.4a}$$

$$\operatorname{curl} \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}, \tag{1.4b}$$

$$\operatorname{div} \mathcal{B} = 0, \tag{1.4c}$$

$$\operatorname{div} \mathcal{D} = \rho, \tag{1.4d}$$

combined with the material laws

$$\mathcal{B} = \mu \mathcal{H}, \quad \mathcal{D} = \varepsilon \mathcal{E} \quad \text{and} \quad \boldsymbol{j}_{\mathrm{c}} = \sigma \mathcal{E}. \tag{1.5}$$

1.4 The Curl–Curl Problem

Several regimes of Maxwell's equations have a common structure, e.g. the magnetostatic problem, the time–harmonic problem and time stepping methods.

This common structure is in our interest to treat in a common framework. To take a look at these types of problems, we have to derive the *vector potential formulation of Maxwell's equations*.

Before we start reformulating, we will consider the de Rham complex.

1.4.1 The de Rham complex

Before we can derive the vector potential formulation of Maxwell's equations we have to do some preparation work. We want to show that the div and curl operator are surjective. Therefore we need the de Rham complex, which can be seen in Figure 1.3.

$$\mathbb{R} \xrightarrow{id} H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl},\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L_2(\Omega) \xrightarrow{0} \{0\}$$

Figure 1.3: The de Rham sequence

The main property of de Rham is the coincidence of ranges and kernels of consecutive (ger.: *aufeinanderfolgend*) operators. For **bounded, simply connected domains** (no wholes or inclusions), the following identities hold

$$\begin{aligned}\ker(\nabla) &= \mathbb{R},\\ \ker(\mathrm{curl}, H(\mathrm{curl},\Omega)) &= \nabla H^1(\Omega),\\ \ker(\mathrm{div}, H(\mathrm{div},\Omega)) &= \mathrm{curl}(H(\mathrm{curl},\Omega)),\\ L_2(\Omega) &= \mathrm{div}(H(\mathrm{div},\Omega)).\end{aligned}$$

In detail, we will need the following two identities in the next subsection:

- div operator is surjective

  For a $\mathcal{B} \in H(\mathrm{div},\Omega)$ it holds:

$$\mathrm{div}\,\mathcal{B} = 0 \quad \overset{\text{de Rham}}{\Rightarrow} \quad \exists \mathcal{A} \in H(\mathrm{curl},\Omega):\ \mathrm{curl}\,\mathcal{A} = \mathcal{B}. \tag{1.6}$$

  Here we show $\nabla H^1(\Omega) \subseteq \ker(\mathrm{curl}, H(\mathrm{curl},\Omega)$. We use the classical results

$$\begin{aligned}\mathrm{curl}\,\nabla v &= 0 \qquad && \forall v \in \mathcal{C}^\infty(\Omega),\\ \mathrm{div}\,\mathrm{curl}\,\underline{v} &= 0 && \forall \underline{v} \in [\mathcal{C}^\infty(\Omega)]^3\,,\end{aligned}$$

  and show, that they also hold in the weak sense.
  Let $w \in H^1(\Omega)$ and $\underline{\tilde{v}} \in [\mathcal{C}_0^\infty(\Omega)]^3$. With the definition of the weak gradient we have

$$\int_\Omega \nabla w \cdot \mathrm{curl}\,\underline{\tilde{v}}\ \mathrm{d}x = -\int_\Omega w \cdot \underbrace{\mathrm{div}\,\mathrm{curl}\,\underline{\tilde{v}}}_{=0}\ \mathrm{d}x = 0.$$

  By computing the weak curl of $\nabla \underline{w} \in [L_2(\Omega)]^3$,

$$\int_\Omega \mathrm{curl}\,\nabla w \cdot \underline{v}\ \mathrm{d}x = \int_\Omega \nabla w \cdot \mathrm{curl}\,\underline{v}\ \mathrm{d}x = 0 \qquad \forall \underline{v} \in [\mathcal{C}_0^\infty(\Omega)]^3\,,$$

  we get that

$$\mathrm{curl}\,\nabla w = 0 \qquad \forall w \in H^1(\Omega).$$

Then, by using the definition of the weak curl we have for a function $u \in H(\mathrm{curl}, \Omega)$

$$\int_\Omega \mathrm{curl}\, \underline{u} \cdot \nabla \tilde{v} \,\mathrm{d}x = \int_\Omega \underline{u} \cdot \mathrm{curl}\, \nabla \tilde{v} \,\mathrm{d}x = 0 \qquad \forall \underline{\tilde{v}} \in \mathcal{C}_0^\infty(\Omega).$$

By computing the weak divergence of the line above we obtain

$$\int_\Omega \mathrm{div}\,\mathrm{curl}\, \underline{u} \cdot v \,\mathrm{d}x = \int_\Omega \mathrm{curl}\, \underline{u} \cdot \nabla v \,\mathrm{d}x = 0 \qquad \forall v \in \mathcal{C}_0^\infty(\Omega)$$

and finally

$$\mathrm{div}\,\mathrm{curl}\, u = 0 \qquad \forall u \in H(\mathrm{curl}\,, \Omega).$$

The other direction can be seen in [5].

- curl operator is surjective

  For an $\mathcal{A} \in H(\mathrm{curl}, \Omega)$ it holds:

$$\mathrm{curl}\, \mathcal{A} = 0 \quad \overset{\text{de Rham}}{\Rightarrow} \quad \exists \varphi \in H^1(\Omega) \text{ with } \int_\Omega \varphi \,\mathrm{d}x = 0 :\ -\nabla\varphi = \mathcal{A}. \tag{1.7}$$

  A proof of this statement can be found in [5] (Theorem 18).

With this knowledge we can start reformulating.

1.4.2 Vector Potential Formulation of Maxwell's Equations

Starting point is Ampère's law in differential form

$$\mathrm{curl}\, \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J},$$

with the material laws

$$\mathcal{D} = \varepsilon \mathcal{E}, \qquad \mathcal{J} = \sigma \mathcal{E} + \boldsymbol{j}_\mathrm{i}, \qquad \mathcal{B} = \mu \mathcal{H} \Rightarrow \mathcal{H} = \mu^{-1} \mathcal{B}.$$

We know that $\mathrm{div}\, \mathcal{B} = 0$ and because of (1.6), there exists a vector potential $\mathcal{A}$ such that

$$\mathcal{B} = \mathrm{curl}\, \mathcal{A}.$$

Plugging in in Ampère's law leads to

$$\mathrm{curl}\, \left(\mu^{-1} \mathrm{curl}\, \mathcal{A}\right) = \varepsilon \frac{\partial \mathcal{E}}{\partial t} + \sigma \mathcal{E} + \boldsymbol{j}_\mathrm{i}. \tag{1.8}$$

Our goal is to get rid of the $\mathcal{E}$. We only want to have one variable $\mathcal{A}$ on the left hand side. Reformulating Faraday's law leads to

$$\operatorname{curl}\mathcal{E} = -\frac{\partial\mathcal{B}}{\partial t} \quad\Longleftrightarrow\quad \operatorname{curl}\mathcal{E} = -\frac{\partial}{\partial t}\operatorname{curl}\mathcal{A} \quad\Longleftrightarrow\quad \operatorname{curl}\left(\mathcal{E} + \frac{\partial\mathcal{A}}{\partial t}\right) = 0.$$

Applying the knowledge that the curl operator is surjective, (1.7), means

$$\operatorname{curl}\left(\mathcal{E} + \frac{\partial\mathcal{A}}{\partial t}\right) = 0 \quad\Rightarrow\quad \exists\,\text{scalar potential}\,\varphi:\ \mathcal{E} = -\nabla\varphi - \frac{\partial\mathcal{A}}{\partial t}.$$

Therefore, (1.8) can be expressed by

$$\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathcal{A}\right) + \varepsilon\frac{\partial^2\mathcal{A}}{\partial t^2} + \sigma\frac{\partial\mathcal{A}}{\partial t} = \boldsymbol{j}_\mathrm{i} - \sigma\nabla\varphi - \varepsilon\frac{\partial\nabla\varphi}{\partial t}.$$

We see, that for any arbitrary scalar function ψ the potentials

$$\tilde{\mathcal{A}} = \mathcal{A} + \nabla\psi,$$
$$\tilde{\varphi} = \varphi - \frac{\partial\psi}{\partial t}$$

satisfy the equation above (plugging in and using Schwarz). By choosing a vector potential $\mathcal{A}^*$ such that

$$\mathcal{A}^* = \mathcal{A} + \int_{t_0}^{t} \nabla\varphi\,\mathrm{d}\,\tilde{t}$$

we get

$$\mathcal{E} = -\frac{\partial\mathcal{A}^*}{\partial t}, \quad \operatorname{curl}\mathcal{A} = \operatorname{curl}\mathcal{A}^*.$$

For convenience, we can introduce

$$\mathcal{E} = -\frac{\partial\mathcal{A}}{\partial t}$$

and finally obtain the *vector potential formulation of Maxwell's equations*

$$\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathcal{A}\right) + \varepsilon\frac{\partial^2\mathcal{A}}{\partial t^2} + \sigma\frac{\partial\mathcal{A}}{\partial t} = \boldsymbol{j}_\mathrm{i}. \tag{1.9}$$

This will be our starting point of analyzing different types of problems in the next subsections.

1.4.3 Time–Harmonic Problem

Many applications in electrical engineering use time–harmonic functions, i.e.

$$\boldsymbol{j}_i(x,t) = \operatorname{Re}\left(\boldsymbol{j}_i(x)e^{i\omega t}\right), \qquad \mathcal{A}(x,t) = \operatorname{Re}\left(\mathcal{A}(x)e^{i\omega t}\right).$$

In this scenario, a time derivation in (1.9) leads to a multiplication with $i\omega$. The time–harmonic vector potential formulation then reads

$$\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathcal{A}\right) + \left(i\omega\sigma - \omega^2\varepsilon\right)\mathcal{A} = \boldsymbol{j}_\mathrm{i},$$

where $\mathcal{A}$ is unknown and $\boldsymbol{j}_\mathrm{i}$ is known. The term $(i\omega\sigma - \omega^2\varepsilon) =: \kappa$ is depending on the type of problem.

1.4.4 Time Stepping Method

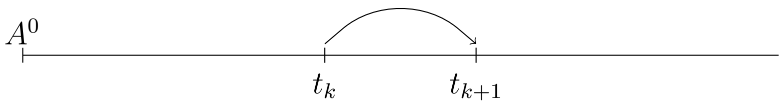
For simplicity we consider a time stepping method for the special regime $\varepsilon = 0$ and $\sigma = 1$. Hence, (1.9) simplifies to

$$\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathcal{A}\right) + \frac{\partial \mathcal{A}}{\partial t} = \boldsymbol{j}_{\mathrm{i}}.$$

Using a simple time stepping method leads to the approximation

$$\frac{\partial \mathcal{A}}{\partial t}\left(t_{k+1}\right) \approx \frac{A^{k+1} - A^{k}}{\Delta t},$$

where $A^k := A(t_k)$.



By applying this approximation we obtain

$$\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathcal{A}^{k+1}\right) + \underbrace{\frac{1}{\Delta t}}_{=:\kappa}\mathcal{A}^{k+1} = \boldsymbol{j}_{\mathrm{i}} + \frac{1}{\Delta t}\mathcal{A}^{k}.$$

where the right hand side is known and $\mathcal{A}^{k+1}$ is unknown.

1.4.5 Result – Curl–Curl Problem

The common structure we have seen in the previous subsections is

$$\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathcal{A}\right) + \kappa\mathcal{A} = \boldsymbol{j}_{\mathrm{i}},$$

where κ is depending on the type of the considered problem, e.g. $(i\omega\sigma - \omega^2\varepsilon)$ for the time–harmonic problem or $1/\Delta t$ for the time stepping method.

The function we are interested in is $\mathcal{A}$. This unknown function is used to calculate the magnetic induction $\mathcal{B} = \operatorname{div}\mathcal{A}$ and the electric field intensity $\mathcal{E} = -\partial\mathcal{A}/\partial t$. For mathematical analysis we will define our unknown function as $\underline{u}$.

In view of deriving a weak formulation and since we know $\boldsymbol{j}_{\mathrm{i}}$, we will define the right hand side as $\underline{f}$.

The starting point of our analysis is the curl–curl problem which reads as

$$\operatorname{curl}\left(\mu^{-1}\operatorname{curl}\underline{u}\right) + \kappa\underline{u} = \underline{f}. \tag{1.10}$$

Chapter 2

Variational Framework

In this chapter we want do derive a weak formulation of the curl–curl problem (1.10). In this work we only consider a special regime. We set $\mu^{-1} = 1$ and $\kappa = 1$, thus our curl–curl problem simplifies to

$$\operatorname{curl}\operatorname{curl}\underline{u} + \underline{u} = \underline{f}, \tag{2.1}$$

for an unknown function $\underline{u}$ and a given right hand side $\underline{f}$. Starting with (2.1), we get the weak formulation by multiplying it with a suitable test function v, integrating the equation over our computational domain Ω, integration by parts of the main part and adding boundary conditions.

At the end of this chapter we want to derive the variational problem: Find $\underline{u} \in H(\operatorname{curl}, \Omega)$ with $\underline{u} \times \underline{n} = 0$, such that

$$\int\limits_{\Omega} \operatorname{curl}\underline{u} \cdot \operatorname{curl}\underline{v} \,\mathrm{d}x + \int\limits_{\Omega} \underline{u} \cdot \underline{v} \,\mathrm{d}x = \int\limits_{\Omega} \underline{f} \cdot \underline{v} \,\mathrm{d}x \qquad \forall \underline{v} \in H(\operatorname{curl}, \Omega), \tag{2.2}$$

with $\underline{v} \times \underline{n} = 0$.

2.1 Integration by Parts

To derive the variational formulation the main part is integrated by parts. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ and outer unit normal vector $\underline{n}$. In (2.1) we can not use the standard integration by parts formula

$$\int\limits_{\Omega} \underline{u} \cdot \frac{\partial \underline{v}}{\partial x_i} \,\mathrm{d}x = -\int\limits_{\Omega} \frac{\partial \underline{u}}{\partial x_i} \cdot \underline{v} + \int\limits_{\partial\Omega} (\underline{u} \cdot n_i) \cdot \underline{v} \,\mathrm{d}s_x. \tag{2.3}$$

Our goal is to obtain an integration by parts formula which can be applied on the main part of (2.1). The following lemma yields the needed integration by parts formula.

Lemma 2.1 (Integration by Parts)**.** *For smooth functions the identity*

$$\int_\Omega \mathit{curl}\,\underline{u} \cdot \underline{v}\; dx = \int_\Omega \underline{u} \cdot \mathit{curl}\,\underline{v}\; dx - \int_{\partial\Omega} (\underline{u} \times \underline{n}) \cdot \underline{v}\; ds_x \tag{2.4}$$

holds.

Proof. We use (2.3) on every term of $\int_\Omega \operatorname{curl}\underline{u} \cdot \underline{v}\,\mathrm{d}x$, i.e.

$$\begin{aligned}
\int_\Omega \operatorname{curl}\underline{u} \cdot \underline{v}\,\mathrm{d}x &= \int_\Omega \nabla \times \underline{u} \cdot \underline{v}\,\mathrm{d}x = \int_\Omega \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ -\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mathrm{d}x \\
&= \int_\Omega \left[\frac{\partial u_3}{\partial x_2} v_1 - \frac{\partial u_2}{\partial x_3} v_1 - \frac{\partial u_3}{\partial x_1} v_2 + \frac{\partial u_1}{\partial x_3} v_2 + \frac{\partial u_2}{\partial x_1} v_3 - \frac{\partial u_1}{\partial x_2} v_3 \right] \mathrm{d}x \\
&= \int_\Omega \left[-u_3 \frac{\partial v_1}{\partial x_2} + u_2 \frac{\partial v_1}{\partial x_3} + u_3 \frac{\partial v_2}{\partial x_1} - u_1 \frac{\partial v_2}{\partial x_3} - u_2 \frac{\partial v_3}{\partial x_1} + u_1 \frac{\partial v_3}{\partial x_2} \right] \mathrm{d}x \\
&\quad - \int_{\partial\Omega} [-(u_3 \cdot n_2) \cdot v_1 + (u_2 \cdot n_3) \cdot v_1 + (u_3 \cdot n_1) \cdot v_2 \\
&\qquad -(u_1 \cdot n_3) \cdot v_2 - (u_2 \cdot n_1) \cdot v_3 + (u_1 \cdot n_2) \cdot v_3]\;\mathrm{d}s_x \\
&= \int_\Omega \left[u_1 \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) + u_2 \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) + u_3 \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \right] \mathrm{d}x \\
&\quad - \int_{\partial\Omega} [(u_2 \cdot n_3 - u_3 \cdot n_2) \cdot v_1 - (u_3 \cdot n_1 - u_1 \cdot n_3) \cdot v_2 \\
&\qquad +(u_1 \cdot n_2 - u_2 \cdot n_1) \cdot v_3]\;\mathrm{d}s_x \\
&= \int_\Omega \underline{u} \cdot \operatorname{curl}\underline{v}\,\mathrm{d}x - \int_{\partial\Omega} (\underline{u} \times \underline{n}) \cdot \underline{v}\,\mathrm{d}s_x.
\end{aligned}$$

□

2.2 Function Space $H(curl, \Omega)$

To derive the variational problem (2.2) we need to introduce a weak curl. To motivate this definition we look at the integration by parts formula (2.4).

Definition 2.2 (Weak Curl)**.** For $\underline{u} \in [L_2(\Omega)]^3$ we call $\operatorname{curl}\underline{u} \in [L_2(\Omega)]^3$ the weak curl of $\underline{u}$, if

$$\int_\Omega \operatorname{curl}\underline{u} \cdot \underline{\varphi}\,\mathrm{d}x = \int_\Omega \underline{u} \cdot \operatorname{curl}\underline{\varphi}\,\mathrm{d}x \qquad \forall \underline{\varphi} \in [\mathcal{C}_0^\infty(\Omega)]^3.$$

This definition of a weak differential operator motivates to define a function space, where all functions have a weak curl. By using this function space we ensure that all expressions in (2.2) are well–defined.

Definition 2.3 ($H(\mathrm{curl},\Omega)$)**.** The space of three–dimensional vector functions with curl in $[L_2]^3$ is defined by

$$H(\mathrm{curl},\Omega) := \left\{ \underline{u} \in [L_2(\Omega)]^3 \ : \ \mathrm{curl}\,\underline{u} \in [L_2(\Omega)]^3 \right\},$$

with the semi–norm and norm

$$|\underline{u}|_{H(\mathrm{curl},\Omega)} := ||\mathrm{curl}\,\underline{u}||_{[L_2(\Omega)]^3},$$

$$||\underline{u}||_{H(\mathrm{curl},\Omega)} := \left[||\underline{u}||^2_{[L_2(\Omega)]^3} + ||\mathrm{curl}\,\underline{u}||^2_{[L_2(\Omega)]^3} \right]^{\frac{1}{2}}.$$

Remark. By a density argument, an equivalent definition of $H(\mathrm{curl},\Omega)$ is given by

$$H(\mathrm{curl},\Omega) := \overline{\mathcal{C}^\infty\left(\overline{\Omega}\right)}^{||\cdot||_{H(\mathrm{curl},\Omega)}},$$

which is by definition a Hilbertspace.

By using Definition 2.3 we have to show, that $[\mathcal{C}^\infty(\overline{\Omega})]^3$ is dense in $H(\mathrm{curl},\Omega)$. For this proof we need a domain Ω with Lipschitz boundary, the existence of smoothing transformations ϕ^ε and commuting smoothing operators. This is carried out in [5].

We see that this is the natural function space for solving the curl–curl problem. The space $H(\mathrm{curl},\Omega)$ has less smoothness than $H^1(\Omega)$, only tangential continuity over material interfaces, as we will see in Section 2.5. This condition holds with the physical properties of electric and magnetic fields.

2.3 Trace Operator and Boundary Conditions

In our integration by parts formula (2.4) we have an integral over $\partial\Omega$. To show that this expression is well defined we need a trace operator. We remember the trace operator in $H^1(\Omega)$,

$$\mathrm{tr}_{\partial\Omega} : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega), \qquad (\mathrm{tr}_{\partial\Omega}\, w)(x) := w(x) \quad \forall x \in \partial\Omega,$$

$$||\mathrm{tr}_{\partial\Omega}\, w||_{H^{\frac{1}{2}}(\partial\Omega)} \leq c_{\mathrm{tr}}\, ||w||_{H^1(\Omega)} \quad \forall w \in H^1(\Omega).$$

We also need the *inverse trace theorem.* Let $g \in H^{\frac{1}{2}}(\partial\Omega)$, then

$$\exists\, w \in H^1(\Omega) \ : \ \mathrm{tr}_{\partial\Omega}\, w = g \quad \text{with} \quad ||w||_{H^1(\Omega)} \leq c_{\mathrm{itr}}||g||_{H^{\frac{1}{2}}(\partial\Omega)}.$$

When speaking about traces in $H(\mathrm{curl},\Omega)$, we define the trace operator as

$$\mathrm{tr}_\tau : H(\mathrm{curl},\Omega) \to [H^{-\frac{1}{2}(\partial\Omega)}]^3, \quad \underline{u} \mapsto \underline{u} \times \underline{n}.$$

Note that $(H^{-\frac{1}{2}}(\partial\Omega))^* = H^{\frac{1}{2}}(\partial\Omega)$. Before we can show the continuity of the trace operator, we need the following estimate.

Lemma 2.4. *For a $\underline{w} \in [H^1(\Omega)]^3$ we have the estimate*

$$||curl\underline{w}||_{[L_2(\Omega)]^3} \leq \sqrt{3}\,||\nabla \underline{w}||_{[L_2(\Omega)]^3}.$$

Proof. We consider the first component of $\operatorname{curl} \underline{w}$. We have

$$||\partial_{x_2} w_3 - \partial_{x_3} w_2||_{L_2(\Omega)} \leq ||\partial_{x_2} w_3||_{L_2(\Omega)} + ||\partial_{x_3} w_2||_{L_2(\Omega)} + ||\partial_{x_1} w_1||_{L_2(\Omega)}$$

Analog we have for the other components

$$||\partial_{x_3} w_1 - \partial_{x_1} w_3||_{L_2(\Omega)} \leq ||\partial_{x_3} w_1||_{L_2(\Omega)} + ||\partial_{x_1} w_3||_{L_2(\Omega)} + ||\partial_{x_2} w_2||_{L_2(\Omega)},$$
$$||\partial_{x_1} w_2 - \partial_{x_2} w_1||_{L_2(\Omega)} \leq ||\partial_{x_1} w_2||_{L_2(\Omega)} + ||\partial_{x_2} w_1||_{L_2(\Omega)} + ||\partial_{x_3} w_3||_{L_2(\Omega)}.$$

Combining the estimates from above, we have that

$$\begin{aligned}
||\operatorname{curl} \underline{w}||^2_{[L_2(\Omega)]^3} &= \sum_{i=1}^{3} ||[\operatorname{curl} \underline{w}]_i||^2_{L_2(\Omega)]^3} \\
&= ||\partial_{x_2} w_3 - \partial_{x_3} w_2||^2_{L_2(\Omega)} + ||\partial_{x_3} w_1 - \partial_{x_1} w_3||^2_{L_2(\Omega)} \\
&\quad + ||\partial_{x_1} w_2 - \partial_{x_2} w_1||^2_{L_2(\Omega)} \\
&\leq \Big(||\partial_{x_2} w_3||_{L_2(\Omega)} + ||\partial_{x_3} w_2||_{L_2(\Omega)} + ||\partial_{x_1} w_1||_{L_2(\Omega)}\Big)^2 \\
&\quad + \Big(||\partial_{x_3} w_1||_{L_2(\Omega)} + ||\partial_{x_1} w_3||_{L_2(\Omega)} + ||\partial_{x_2} w_2||_{L_2(\Omega)}\Big)^2 \\
&\quad + \Big(||\partial_{x_1} w_2||_{L_2(\Omega)} + ||\partial_{x_2} w_1||_{L_2(\Omega)} + ||\partial_{x_3} w_3||_{L_2(\Omega)}\Big)^2 \\
&\leq 3\Big(||\partial_{x_1} w_1||^2_{L_2(\Omega)} + ||\partial_{x_2} w_1||^2_{L_2(\Omega)} + ||\partial_{x_3} w_1||^2_{L_2(\Omega)} \\
&\quad + ||\partial_{x_1} w_2||^2_{L_2(\Omega)} + ||\partial_{x_2} w_2||^2_{L_2(\Omega)} + ||\partial_{x_3} w_2||^2_{L_2(\Omega)} \\
&\quad + ||\partial_{x_1} w_3||^2_{L_2(\Omega)} + ||\partial_{x_2} w_3||^2_{L_2(\Omega)} + ||\partial_{x_3} w_3||^2_{L_2(\Omega)}\Big) \\
&= 3\,||\nabla \underline{w}||^2_{[L_2(\Omega)]^3}.
\end{aligned}$$

□

With this curl–estimate we can show, that $||\operatorname{tr}_\tau \underline{u}||_{[H^{-\frac{1}{2}}(\partial\Omega)]^3} \leq c_{\mathrm{tr}} ||\underline{u}||_{H(\mathrm{curl},\Omega)}$. For $\underline{u} \in [\mathcal{C}^\infty(\overline{\Omega})]^3$ we have

$$\begin{aligned}
||\operatorname{tr}_\tau \underline{u}||_{[H^{-\frac{1}{2}}(\partial\Omega)]^3} &= \sup_{0 \neq \underline{v} \in [H^{\frac{1}{2}}(\partial\Omega)]^3} \frac{\langle \underline{u} \times \underline{n}, \underline{v} \rangle}{||\underline{v}||_{[H^{\frac{1}{2}}(\partial\Omega)]^3}} = \sup_{0 \neq \underline{v} \in [H^{\frac{1}{2}}(\partial\Omega)]^3} \frac{\int_\Omega (\underline{u} \times \underline{n}) \cdot \underline{v} \,\mathrm{d}s_x}{||\underline{v}||_{[H^{\frac{1}{2}}(\partial\Omega)]^3}} \\
&\overset{(2.3)}{=} \sup_{0 \neq \underline{v} \in [H^{\frac{1}{2}}(\partial\Omega)]^3} \frac{\int_\Omega \underline{u} \cdot \operatorname{curl} \underline{v} \,\mathrm{d}x - \int_\Omega \operatorname{curl} \underline{u} \cdot \underline{v} \,\mathrm{d}x}{||\underline{v}||_{[H^{\frac{1}{2}}(\partial\Omega)]^3}} \\
&\leq \sup_{0 \neq \underline{w} \in [H^1(\Omega)]^3} \frac{\int_\Omega \underline{u} \cdot \operatorname{curl} \underline{w} \,\mathrm{d}x - \int_\Omega \operatorname{curl} \underline{u} \cdot \underline{w} \,\mathrm{d}x}{c_{\mathrm{itr}}^{-1} ||\underline{w}||_{[H^1(\Omega)]^3}}
\end{aligned}$$

$$\leq c_{\text{itr}} \sup_{0\neq \underline{w}\in[H^1(\Omega)]^3} \frac{\int_\Omega |\underline{u}\cdot \operatorname{curl}\underline{w}|\, \mathrm{d}x + \int_\Omega |\operatorname{curl}\underline{u}\cdot\underline{w}|\, \mathrm{d}x}{||\underline{w}||_{[H^1(\Omega)]^3}}$$

$$\leq c_{\text{itr}} \sup_{0\neq \underline{w}\in[H^1(\Omega)]^3} \frac{||\underline{u}||_{[L_2(\Omega)]^3}||\operatorname{curl}\underline{w}||_{[L_2(\Omega)]^3} + ||\operatorname{curl}\underline{u}||_{[L_2(\Omega)]^3}||\underline{w}||_{[L_2(\Omega)]^3}}{||\underline{w}||_{[H^1(\Omega)]^3}}$$

$$\leq c_{\text{itr}} \sup_{0\neq \underline{w}\in[H^1(\Omega)]^3} \frac{\sqrt{3}\,||\underline{u}||_{[L_2(\Omega)]^3}||\nabla\underline{w}||_{[L_2(\Omega)]^3} + ||\operatorname{curl}\underline{u}||_{[L_2(\Omega)]^3}||\underline{w}||_{[L_2(\Omega)]^3}}{||\underline{w}||_{[H^1(\Omega)]^3}}$$

$$\leq \sqrt{3}\, c_{\text{itr}} \sup_{0\neq \underline{w}\in[H^1(\Omega)]^3} \frac{\left[||\underline{u}||^2_{[L_2(\Omega)]^3} + ||\operatorname{curl}\underline{u}||^2_{[L_2(\Omega)]^3}\right]^{\frac{1}{2}}\left[||\nabla\underline{w}||^2_{[L_2(\Omega)]^3} + ||\underline{w}||^2_{[L_2(\Omega)]^3}\right]^{\frac{1}{2}}}{\left[||\underline{w}||^2_{[L_2(\Omega)]^3} + ||\nabla\underline{w}||^2_{[L_2(\Omega)]^3}\right]^{\frac{1}{2}}}$$

$$\leq \sqrt{3}\, c_{\text{itr}}||\underline{u}||_{H(\operatorname{curl},\Omega)}.$$

Thus, the trace on $\partial\Omega$ is well defined. Now we need to find useful boundary conditions. When deriving our weak formulation we will consider the integral

$$\int_{\partial\Omega} (\operatorname{curl}\underline{u}\times\underline{n})\cdot\underline{v}\, \mathrm{d}s_x.$$

We can write

$$(\operatorname{curl}\underline{u}\times\underline{n})\cdot\underline{v} = (\operatorname{curl}\underline{u}\times\underline{n})\cdot((\underline{v}\times\underline{n})\times\underline{n}),$$

where

- $\operatorname{curl}\underline{u}\times\underline{n}$ is a natural boundary,
- $((\underline{v}\times\underline{n})\times\underline{n})$ is an essential boundary. Moreover, $\underline{v}\times\underline{n}$ is the tangential component of $\underline{v}$.

In this thesis we will use the boundary conditions $\underline{u}\times\underline{n} = 0$ and $\underline{v}\times\underline{n} = 0$.

2.4 Weak Formulation

We integrate (2.1) over the computational domain Ω and multiply it with a proper test function $\underline{v}$, i.e.

$$\int_\Omega \operatorname{curl}(\operatorname{curl}\underline{u})\cdot\underline{v}\, \mathrm{d}x + \int_\Omega \underline{u}\cdot\underline{v}\, \mathrm{d}x = \int_\Omega \underline{f}\cdot\underline{v}\, \mathrm{d}x.$$

Using integration by parts for the main part leads to

$$\int_\Omega \operatorname{curl}(\operatorname{curl}\underline{u})\cdot\underline{v}\, \mathrm{d}x = \int_\Omega \operatorname{curl}\underline{u}\cdot\operatorname{curl}\underline{v}\, \mathrm{d}x - \int_{\partial\Omega} (\operatorname{curl}\underline{u}\times\underline{n})\cdot\underline{v}\, \mathrm{d}s_x.$$

Thus, we obtain the variational formulation

$$\int_\Omega \operatorname{curl} \underline{u} \cdot \operatorname{curl} \underline{v} \, \mathrm{d}x + \int_\Omega \underline{u} \cdot \underline{v} \, \mathrm{d}x = \int_\Omega \underline{f} \cdot \underline{v} \, \mathrm{d}x + \int_{\partial\Omega} (\operatorname{curl} \underline{u} \times \underline{n}) \cdot \underline{v} \, \mathrm{ds}_x,$$

and including our boundary conditions finally leads to:

Find $\underline{u} \in H(\operatorname{curl}, \Omega)$ with $\underline{u} \times \underline{n} = 0$, such that

$$\int_\Omega \operatorname{curl} \underline{u} \cdot \operatorname{curl} \underline{v} \, \mathrm{d}x + \int_\Omega \underline{u} \cdot \underline{v} \, \mathrm{d}x = \int_\Omega \underline{f} \cdot \underline{v} \, \mathrm{d}x. \tag{2.5}$$

for all $\underline{v} \in H(\operatorname{curl}, \Omega)$ with $\underline{v} \times \underline{n} = 0$.
We finally obtained a weak formulation of Maxwell's equations.

With the *lemma of Lax–Milgram* we get, that there is a unique solution for (2.5). Since the functional

$$f(\underline{v}) = \int_\Omega \underline{f} \cdot \underline{v} \, \mathrm{d}x \overset{C.S.}{\leq} ||\underline{f}||_{[L_2(\Omega)]^3} ||\underline{v}||_{[L_2(\Omega)]^3} \leq ||\underline{f}||_{[L_2(\Omega)]^3} ||\underline{v}||_{H(\operatorname{curl},\Omega)},$$

is bounded, and $a(u, v)$ is bounded and coercive

$$\begin{aligned} a(\underline{v}, \underline{v}) &\geq c_1^a ||\underline{v}||^2_{H(\operatorname{curl},\Omega)} && \forall v \in V, \\ a(\underline{u}, \underline{v}) &\leq c_2^a ||\underline{u}||_{H(\operatorname{curl},\Omega)} ||\underline{v}||_{H(\operatorname{curl},\Omega)} && \forall u, v \in V, \end{aligned}$$

with $c_1^a = c_2^a = 1$, we can apply Lax–Milgram.

2.5 Interface Condition

In this section we want to find a condition for $\underline{u}$ on the interface $\Gamma = \Omega_1 \cap \Omega_2$, such that for $\underline{u}_1 \in H(\operatorname{curl}, \Omega_1)$ and $\underline{u}_2 \in H(\operatorname{curl}, \Omega_2)$ we can write

$$\underline{u}(\underline{x}) = \begin{Bmatrix} \underline{u}_1(\underline{x}) & \text{for } \underline{x} \in \overline{\Omega}_1 \\ \underline{u}_2(\underline{x}) & \text{for } \underline{x} \in \overline{\Omega}_2 \end{Bmatrix} \in H(\operatorname{curl}, \Omega). \tag{2.6}$$

The decomposition into two domains is illustrated in Figure 2.1, where $\underline{n}$ is an outer unit normal vector pointing from the region Ω_1 to Ω_2. The outer unit normal vector for Ω_2 is $-\underline{n}$.
The following lemma shows the wanted *interface condition* (2.7), such that we can combine two functions which are joined via an interface Γ.

Lemma 2.5. *Let $\underline{u}_1 \in H(curl, \Omega_1)$, $\underline{u}_2 \in H(curl, \Omega_2)$ and $\underline{u}$ is defined by* (2.6)*. Let $\underline{n}$ be an outer unit normal vector from Ω_1. If*

$$\underline{u}_1 \times \underline{n} = \underline{u}_2 \times \underline{n} \text{ on } \Gamma, \tag{2.7}$$

then $\underline{u} \in H(curl, \Omega)$.

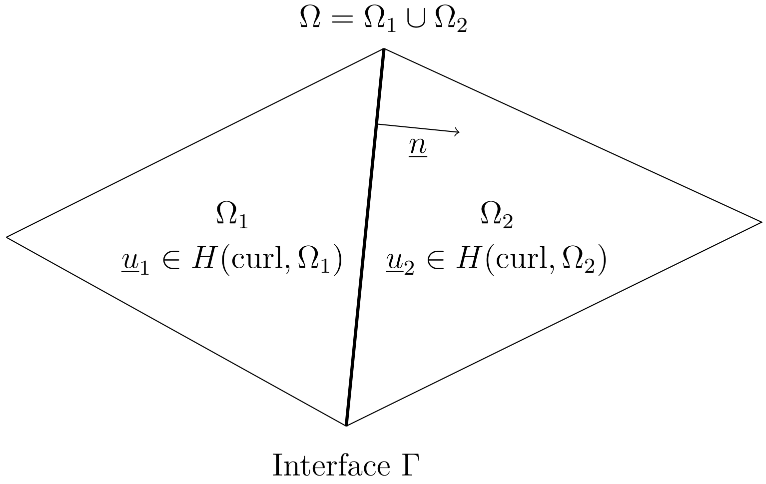


Figure 2.1: Schematic of finding the interface condition for decomposition

Proof. We have to show, that

$$\int_\Omega \mathrm{curl}\, \underline{u} \cdot \underline{v} \,\mathrm{d}x = \int_\Omega \underline{u} \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x \qquad \forall \underline{v} \in [\mathcal{C}_0^\infty(\Omega)]^3,$$

then $\underline{u} \in H(\mathrm{curl}, \Omega)$. For $\underline{v} \in [\mathcal{C}_0^\infty(\Omega)]^3$ it holds

$$\begin{aligned}
\int_\Omega \mathrm{curl}\, \underline{u} \cdot \underline{v} \,\mathrm{d}x &= \int_{\Omega_1 \cup \Omega_2} \mathrm{curl}\, \underline{u} \cdot \underline{v} \,\mathrm{d}x \\
&= \int_{\Omega_1} \mathrm{curl}\, \underline{u}_1 \cdot \underline{v} \,\mathrm{d}x + \int_{\Omega_2} \mathrm{curl}\, \underline{u}_2 \cdot \underline{v} \,\mathrm{d}x \\
&= \int_{\Omega_1} \underline{u}_1 \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x - \int_\Gamma (\underline{u}_1 \times \underline{n}) \cdot \underline{v} \,\mathrm{d}x \\
&\quad + \int_{\Omega_2} \underline{u}_2 \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x - \int_\Gamma -(\underline{u}_2 \times \underline{n}) \cdot \underline{v} \,\mathrm{d}x \\
&= \int_{\Omega_1} \underline{u}_1 \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x + \int_{\Omega_2} \underline{u}_2 \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x - \int_\Gamma (\underbrace{(\underline{u}_1 \times \underline{n}) - (\underline{u}_2 \times \underline{n})}_{=0}) \cdot \underline{v} \,\mathrm{d}x \\
&= \int_{\Omega_1} \underline{u}_1 \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x + \int_{\Omega_2} \underline{u}_2 \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x \\
&= \int_{\Omega_1 \cup \Omega_2} \underline{u} \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x \\
&= \int_\Omega \underline{u} \cdot \mathrm{curl}\, \underline{v} \,\mathrm{d}x.
\end{aligned}$$

□

Chapter 3

Finite Elements for Maxwell Equations

When we speak about finite elements, we do this in the sense of CIARLET in [2]. In order to apply the Galerkin method we have to construct finite element spaces X_h of X, where X is for example $H^1(\Omega), H(\text{curl}, \Omega)$ or $H(\text{div}, \Omega)$. We have to consider three basic aspects. First, a *triangulation* over the computational domain Ω is needed. Secondly, after having fixed a finite element space X_h, we want to define the *local finite–dimensional spaces*

$$X_T := \{v_h|_T \,:\, v_h \in X_h\}, \tag{3.1}$$

which contain polynomials. The third basic aspect of the finite element method is, that there exists at least one *canonical basis of* X_h such that the corresponding basis functions have small support.

Here we want to find global functionals, which form a basis for X_h and local element spaces X_T. Then the finite element spaces X_h are fixed. When finding this spaces we want to derive a discrete exact sequence – cf. the de Rham complex.

We define nodal interpolation operators which satisfy a commuting property. By considering this condition we will create an exact sequence. To satisfy $X_h \subseteq X$, we have to show, that a function in X_h is also contained in the Sobolev space X.

After that we will take a look on transformations. By using this principle, many calculations can be done a–priori on a reference element. By defining such transformations one has to take care, that the global functionals are preserved.

Finally, an interpolation error estimate is formulated.

3.1 Triangulation and Finite Element

When speaking about finite elements, we need a triangulation of our computational domain $\Omega \subseteq \mathbb{R}^3$. Here we assume that Ω is a *bounded polyhedral domain with Lipschitz boundary.*

Definition 3.1. A *triangulation* resp. a mesh $\mathcal{T}_h$ is a finite non–overlapping subdivision of Ω into elements T_i of simple geometry.

A triangulation is called *admissible*, if

- the elements are non–overlapping, i.e.

$$\text{interior}(T_i) \cap \text{interior}(T_j) = \emptyset \text{ for } i \neq j.$$

- the triangulation $\mathcal{T}_h$ is a covering of Ω, i.e.

$$\bigcup_{T_i \in \mathcal{T}_h} T_i = \overline{\Omega}.$$

- the intersection of two different elements is either empty, or a vertex, or an edge or a face of both elements.

After having the most characteristic aspect of the finite element method, the triangulation, we can define a finite element.

Definition 3.2 (Finite Element)**.** The triple (T, X_T, B_T) is called a finite element where

- $T \subset \mathbb{R}^n$ is called the element domain (bounded closed set with non–empty interior and piecewise smooth boundary),
- X_T is the space of shape functions (finite–dimensional space of functions on T),
- $B_T = \{f_1^T, f_2^T, \ldots, f_k^T\}$ is the set of nodal variables.

Remark. The set of nodal variables are also called *degrees of freedom* (dofs).

The main effort by constructing finite elements is to verify the unisolvence. We show this by an equivalent property of the basis property:

$$\text{If } v \in X_T \text{ with } f_i^T v = 0 \text{ for all } i, \text{ then } v \equiv 0.$$

To describe a $v \in X_T$ we define a basis for the space. This basis is called the nodal basis.

Definition 3.3 (Nodal Basis)**.** Let (T, X_T, B_T) be a finite element. The basis

$$\{\varphi_1, \varphi_2, \ldots, \varphi_k\} \text{ of } X_T,$$

which is dual to the set of nodal variables B_T (i.e. $f_i(\varphi_j) = \delta_{ij}$) is called the *nodal basis* of X_T.

3.2 Discrete Exact Sequence

Looking at (3.1) we see, that having the global functionals which form a basis for X_h, and defining the local element spaces X_T, fixes the finite element space X_h. In this section we want to find such functionals and the local element spaces. Furthermore, we want to derive a commuting diagram which is exact, i.e. that the image of the previous differential operator is the kernel of the following one.

$$\begin{array}{ccccccc} H^1(\Omega) & \xrightarrow{\nabla} & H(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L_2(\Omega) \\ \cup & & \cup & & \cup & & \| \\ W & & V & & Q & & S \\ \mathrm{I}^{\nabla}\downarrow & & \mathrm{I}^{\mathrm{curl}}\downarrow & & \mathrm{I}^{\mathrm{div}}\downarrow & & \mathrm{I}^{0}\downarrow \\ W_h & \xrightarrow{\nabla} & V_h & \xrightarrow{\mathrm{curl}} & Q_h & \xrightarrow{\mathrm{div}} & S_h \end{array}$$

To get exactness we assume, that the sequence

$$\begin{array}{ccccccc} H^1(\Omega) & \xrightarrow{\nabla} & H(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & L_2(\Omega) \\ \cup & & \cup & & \cup & & \cup \\ \mathcal{C}^\infty(\overline{\Omega}) & \xrightarrow{\nabla} & [\mathcal{C}^\infty(\overline{\Omega})]^3 & \xrightarrow{\mathrm{curl}} & [\mathcal{C}^\infty(\overline{\Omega})]^3 & \xrightarrow{\mathrm{div}} & \mathcal{C}^\infty(\overline{\Omega}) \end{array}$$

is exact. We know that the lower line of the sequence is exact and because of density and by constructing commuting smoothing operators also the higher line – cf. [5].

3.2.1 Interpolation Operator

In this work we only consider nodal interpolation operators. In the sequence we want to derive, we have appropriate subspaces W, V, Q, S of the corresponding function spaces. We need them, because our operators are not well defined on the whole function space (e.g. point evaluation is not well defined in H^1).

Definition 3.4 (Nodal Interpolation Operator)**.** Let X be a space and X_h the corresponding FE space of X. The nodal interpolation operator $I^X : X \to X_h$ is defined by

$$I^X(x) := \sum_{i=1}^{N} f_i(x)\varphi_i,$$

where f_i are the global functionals and φ_i is the nodal basis of X.

Remark. An alternative are *Clémént type operators* which involve non local averaging over patches. This type of operators require less smoothness.

Lemma 3.5. *The Nodal Interpolation Operator is a projection, i.e.*

$$I^X\left(I^X(x)\right) = I^X(x).$$

Proof. The nodal basis $\{\varphi_i\}_{i=1}^N$ can be associated with the global functionals $\{f_j\}_{j=1}^N$ with the property

$$f_j(\varphi_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases} \qquad \text{for } i, j = 1, \dots, N.$$

Moreover it holds

$$f_j\left(I^X(x)\right) = \sum_{i=1}^N f_i(x) f_j(\varphi_i) = f_j(x).$$

That means

$$f_j\left(I^X(x)\right)\varphi_j = f_j(x)\varphi_j \quad \Longleftrightarrow \quad \sum_{j=1}^N f_j\left(I^X(x)\right)\varphi_j = \sum_{j=1}^N f_j(x)\varphi_j,$$

and by using the definition of the nodal interpolation operator the result follows. □

Next, we want to find a condition which says, when our nodal interpolation operators commute. Considering this condition helps us to derive an exact sequence.

3.2.2 Commuting Property

We pick out two spaces and investigate the interpolation operators and a differential operator D.

$$\begin{array}{ccc} A & \xrightarrow{D} & B \\ {\scriptstyle \mathrm{I}^A}\downarrow & & {\scriptstyle \mathrm{I}^B}\downarrow \\ A_h & \xrightarrow{D} & B_h \end{array}$$

We say that the interpolation operators commute, if

$$\boxed{D\,\mathrm{I}^A a = \mathrm{I}^B\, Da \qquad \forall a \in A.} \tag{3.2}$$

An equivalent condition when the interpolation operators commute can be obtained by using Lemma 3.5.

$$\begin{aligned} D\,\mathrm{I}^A a = \mathrm{I}^B\, Da & \qquad \forall a \in A, \\ & \Updownarrow \\ \mathrm{I}^B D\,\mathrm{I}^A a = \mathrm{I}^B\, Da & \qquad \forall a \in A, \\ & \Updownarrow \\ \mathrm{I}^B(D\,\underbrace{(a - \mathrm{I}^A)}_{=:e\ldots\text{error}}) = 0 & \qquad \forall a \in A \end{aligned}$$

Lemma 3.6 (Exactness of Sequence)**.** *Let*

$$A \xrightarrow{D_1} B \xrightarrow{D_2} C$$

be an exact sequence, i.e. $\ker(D_2, B) = D_1(A)$*. If the interpolation operators commute, then also the lower sequence of*

$$\begin{array}{ccccc} A & \xrightarrow{D_1} & B & \xrightarrow{D_2} & C \\ {\scriptstyle I^A}\big\downarrow & & {\scriptstyle I^B}\big\downarrow & & {\scriptstyle I^C}\big\downarrow \\ A_h & \xrightarrow{D_1} & B_h & \xrightarrow{D_2} & C_h \end{array}$$

is exact, i.e. $\ker(D_2, B_h) = D_1(A_h)$*.*

Proof. "$\supseteq$": Since $B_h \subseteq B$ we have

$$\ker(D_2, B_h) = \{b_h \in B_h \,:\, D_2 b_h = 0\} \subseteq \{b \in B \,:\, D_2 b = 0\} = \ker(D_2, B).$$

For an $a_h \in A_h$ we know $D_1 a_h \in B_h$. By using the exactness of the upper sequence we obtain

$$D_1 a_h \in D_1(A_h) \subseteq D_1(A) = \ker(D_2, B).$$

Since $D_1 a_h \in \ker(D_2, B)$ it follows $D_1 a_h \in \ker(D_2, B_h)$ which implies $D_1(A_h) \subseteq \ker(D_2, B_h)$.
"$\subseteq$": Let $b_h \in \ker(D_2, B_h)$. Because of $\ker(D_2, B_h) \subseteq \ker(D_2, B)$

$$\exists\, a \in A \,:\, D_1 a = b_h.$$

By using (3.2) it follows

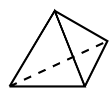
$$b_h = I^B b_h = I^B D_1 a = D_1 I^A a = D_1 a_h.$$

Hence, $\ker(D_2, B_h) \subseteq D_1(A_h)$. $\square$

3.2.3 Global Functionals

One step to fix our finite element space X_h is to define the global functionals. If we consider the commuting property, we ensure that the discrete sequence will be exact.
We start on the end of the sequence, i.e. we define the global functionals for $X_h^0 \subseteq L^2(\Omega)$. The functional we use is given by

$$f_T^0(s) := \int_T s(x)\,\mathrm{d}x \quad \text{for all tetrahedra } T.$$



T

The interpolation operator is given by

$$I^0(s) := \sum_{T \in \mathcal{T}} f_T^0(s) \varphi_T^0,$$

where $\varphi^0_{T_i}$ are the nodal basis functions and $\mathcal{T}$ the set of tetrahedra. It is well known that

$$I^0(s) = 0 \quad \Leftrightarrow \quad f^0_T(s) = 0 \quad \forall T \in \mathcal{T}.$$

Therefore, the commuting property changes to

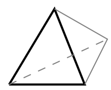
$$f^0_T\Big(\operatorname{div}(\underbrace{q - I^{\mathrm{div}}}_{=:e^{\mathrm{div}}})\Big) = 0 \qquad \forall T \in \mathcal{T}.$$

Using Gauss's theorem we obtain

$$0 = \int_T \operatorname{div} e^{\mathrm{div}} \,\mathrm{d}x = \int_{\partial T} e^{\mathrm{div}} \cdot n \,\mathrm{d}s_x = \sum_{F\in\mathcal{F}} \int_F e^{\mathrm{div}} \cdot n_{F_k} \,\mathrm{d}s_x,$$

where $\mathcal{F}$ is the set of faces of T. This motivates to define the functional for the previous subspace $X^{\mathrm{div}}_h \subset H(\mathrm{div}, \Omega)$,

$$f^{\mathrm{div}}_F(q) := \int_F q(x) \cdot n_F \,\mathrm{d}s_x,$$



where F_k is a face of the tetrahedron and n denotes the unit outer normal vector. Now we look at the condition

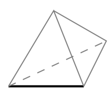
$$f^{\mathrm{div}}_F\Big(\operatorname{curl}(\underbrace{v - I^{\mathrm{curl}}}_{=:e^{\mathrm{curl}}})\Big) = 0 \qquad \forall F \in \mathcal{F}.$$

Using Stokes' theorem we obtain

$$0 = \int_F \operatorname{curl} e^{\mathrm{curl}} \,\mathrm{d}x = \int_{\partial F} e^{\mathrm{curl}} \cdot \tau \,\mathrm{d}\ell_x = \sum_{E\in\mathcal{E}} \int_E e^{\mathrm{curl}} \cdot \tau_E \,\mathrm{d}\ell_x,$$

where $\mathcal{E}$ is the set of edges of F. This motivates to define the functional for the previous subspace $X^{\mathrm{curl}}_h \subset H(\mathrm{curl}, \Omega)$,

$$f^{\mathrm{curl}}_E(v) := \int_E v(x) \cdot \tau_E \,\mathrm{d}\ell_x,$$



where E is an edge of the face. Now we consider the condition

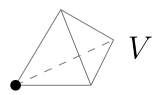
$$f^{\mathrm{curl}}_E\Big(\nabla(\underbrace{w - I^{\mathrm{grad}}}_{=:e^{\mathrm{grad}}})\Big) = 0 \qquad \forall E \in \mathcal{E}.$$

Using the fact that $\nabla e^{\mathrm{grad}} \cdot \tau$ is the tangential derivative and the *fundamental theorem of calculus* we obtain

$$0 = \int_E \nabla e^{\mathrm{grad}} \cdot \tau_E \,\mathrm{d}\ell_x = e^{\mathrm{grad}}(V_j) - e^{\mathrm{grad}}(V_i),$$

where V_j, V_i are the two outer vertices of the edge E. The point evaluation is the functional used in $X_h^{\text{grad}} \subset H^1(\Omega)$, i.e.

$$f_V^{\text{grad}}(w) := w(V),$$

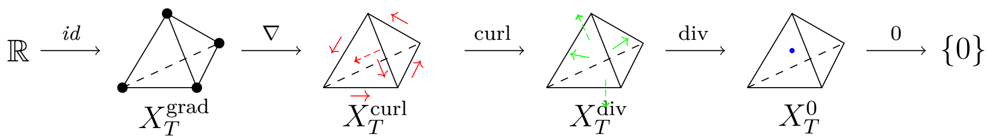


where V is a vertex of an edge.

Since we defined all global functionals respectively the degrees of freedom in the discrete spaces, we consider the local element spaces X_T to fix the finite element spaces X_h next.

3.2.4 Local Element Spaces

The last step to fix X_h is to define the local element spaces X_T. With our choice we will also get, that the discrete sequence is exact. Consider the following sequence:



We know that the local element spaces for $X_h^{\text{grad}} \subset H^1(\Omega)$ and $X_h^0 \subset L^2(\Omega)$ are given by

$$X_T^{\text{grad}} := \left\{ c_1 + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \,:\, c_1, a_1, a_2, a_3 \in \mathbb{R} \right\} \quad \text{with} \quad \dim X_T^{\text{grad}} = 4$$

respectively

$$X_T^0 := \{ c_1 \,:\, c_1 \in \mathbb{R} \} \quad \text{with} \quad \dim X_T^0 = 1.$$

With this local element spaces we have fixed X_h^{grad} and X_h^0. Next, we want to find X_T^{curl} and X_T^{div}.
We know that the gradient of a linear function, which is contained in X_T^{grad}, is constant. Thus we look for linear functions such that the kernel of the curl is given only by constants. Hence we define

$$X_T^{\text{curl}} := \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,:\, c_1, c_2, c_3, a_1, a_2, a_3 \in \mathbb{R} \right\} \quad \text{with} \quad \dim X_T^{\text{curl}} = 6.$$

Remark. X_T^{curl} in combination with f_E^{curl} is also called the *Nédélec element of first kind of order zero.*

Therefore we have for a $v_T \in X_T^{\text{curl}}$ that

$$\operatorname{curl} v_T = \operatorname{curl} \left(\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) = 2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

that means

$$\operatorname{curl} v_T = 0 \quad \Rightarrow \quad v_T = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in [\mathbb{P}_0(T)]^3 .$$

Again for $X_h^{\text{div}} \subset H(\text{div}, \Omega)$, we look for functions such that the kernel of the divergence of these functions is given only by constants. Thus we introduce

$$X_T^{\text{div}} := \left\{ \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + a_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \; : \; c_1, c_2, c_3, a_1 \in \mathbb{R} \right\} \quad \text{with} \quad \dim X_T^{\text{div}} = 4.$$

Remark. X_T^{div} in combination with f_E^{div} is also called the *Raviart–Thomas element of first kind of order zero.* Although Nédélec was the first one in 3d, these finite elements are called after Pierre–Arnaud *Raviart* and Jean–Marie *Thomas*, who found them in 2d first.

Considering a $q_T \in X_T^{\text{div}}$ provides

$$\operatorname{div} q_T = \operatorname{div} \left(\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + a_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) = 3a_1,$$

that means

$$\operatorname{div} q_T = 0 \quad \Rightarrow \quad q_T = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \in [\mathbb{P}_0(T)]^3 .$$

Summarizing we have

$$\nabla X_T^{\text{grad}} \subseteq X_T^{\text{curl}}, \qquad \operatorname{curl} X_T^{\text{curl}} \subseteq X_T^{\text{div}}, \qquad \operatorname{div} X_T^{\text{div}} = X_T^0.$$

To get exactness of the discrete sequence we have to check, if the derived functionals determine a function on the local element spaces. For X_T^0 and X_T^{grad} this is well known, so we only consider X_T^{curl} and X_T^{div}.
In other words we have to check, when the global functionals are zero on all edges or faces, then the local function itself has to be zero. Therefore let $q_T \in X_T^{\text{div}}$ and $f_F^{\text{div}}(q_T) = 0$ for all faces of a tetrahedron. Computing

$$\int_T \operatorname{div} q_T \, \mathrm{d}x = \int_{\partial T} q_T \cdot n \, \mathrm{d}s_x = \sum_{k=1}^{4} \int_{F_k} q_T \cdot n \, \mathrm{d}s_x = \sum_{k=1}^{4} f_F^{\text{div}}(q_T) = 0$$

implies $\operatorname{div} q_T = 0$, thus q_T has to be constant. Because of $f_F^{\text{div}}(q_T) = 0$ for all faces of a tetrahedron, $q_T = 0$.
Let $v_T \in X_T^{\text{curl}}$ and $f_E^{\text{curl}}(v_T) = 0$ for all edges of a tetrahedron. Considering

$$\int_T \operatorname{curl} v_T \, \mathrm{d}x = \int_{\partial T} v_T \cdot \tau \, \mathrm{d}\ell_x = \sum_{j=1}^{6} \int_{E_j} v_T \cdot \tau \, \mathrm{d}\ell_x = \sum_{j=1}^{6} f_E^{\text{curl}}(v_T) = 0$$

implies $\operatorname{curl} v_T = 0$, thus v_T has to be constant. Again, all functionals of v_T are zero and therefore $v_T = 0$.

3.2.5 Conformity of Global Functions

We know that our discrete sequence is exact and our global functionals are a basis for global discrete functions. The last step is to check, if the global functions are contained in $H(\mathrm{curl},\Omega)$ or $H(\mathrm{div},\Omega)$. For the other spaces this is clear.
So, for these two spaces we have to check, if the traces are continuous. Hence we will show that the functionals restricted to a face F uniquely determine the trace on the face F.
For $H(\mathrm{div},\Omega)$ we need that the normal component is continuous. Let $q_T \in X_T^{\mathrm{div}}$. For a face F with unit normal n we have

$$q_T \cdot n = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \cdot n + a_1 \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot n}_{=||x||_{\ell^2}||n||_{\ell^2}\cos\theta} \in \mathbb{P}_0(F),$$

where θ is the angle between x and n. Now let $f_F^{\mathrm{div}}(q_T) = 0$ for a face $F \in \mathcal{F}$. Since $q_T \cdot n$ is constant and

$$\int_F q_T \cdot n \, \mathrm{ds}_x = 0 \qquad \Longrightarrow \qquad q_T \cdot n = 0 \text{ on } F.$$

So we have $X_h^{\mathrm{div}} \subset H(\mathrm{div},\Omega)$. Consider $H(\mathrm{curl},\Omega)$, we need that the tangential trace is continuous. Let $v_T \in X_T^{\mathrm{curl}}$. Then we have for an edge with corresponding tangential vector τ that

$$v_T \cdot \tau = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \cdot \tau + \left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right] \cdot \tau = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \cdot \tau + \left[\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times V \right] \cdot \tau \in \mathbb{P}_0(E),$$

where V is a vertex on the edge E. Again, let all functionals belonging to the face F be zero, i.e. $f_E^{\mathrm{curl}}(v_T) = 0$ for all edges. Thus we have

$$0 = f_E^{\mathrm{curl}}(v_T) = \int_E v_T \cdot \tau \, \mathrm{d}\ell_x \qquad \Longrightarrow \qquad v_T \cdot \tau = 0 \text{ on } E.$$

Finally, we have to consider the trace of v_T in $H(\mathrm{curl})$, i.e. $v_T \times n$. Therefore let V be a vertex of the face F with two tangential vectors τ_1, τ_2 leaving V. Then by using the *vector triple product* (cf. [5]), we obtain

$$v_T \times n = v_T \times (\tau_1 \times \tau_2) = \tau_1(v_T \cdot \tau_2) - \tau_2(v_T \cdot \tau_1) = 0. \tag{3.3}$$

Since $v_T \in X_T^{\mathrm{curl}}$ is linear, also $v_T \times n|_F$ is linear. By applying (3.3) for each node of the face we have $v_T \times n|_F = 0$ and so we have shown that $X_h^{\mathrm{curl}} \subset H(\mathrm{curl},\Omega)$.

Summarizing, we have that the discrete finite element spaces are conforming.

3.3 Transformation

The mapping trick plays an important role for the FEM. The physical elements are seen as transformations of a local reference element of simple shape – here a tetrahedron. This construction has the big advantage, that many computations can be done a–priori on the reference element and are afterwards transformed to the physical element – as illustrated in Figure 3.1. We only consider affine transformations

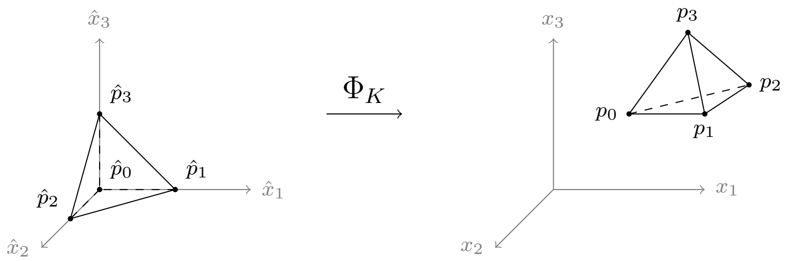


Figure 3.1: Mapping Trick

which are continuously differentiable, bijective and onto maps, that means

$$\Phi_K : \hat{K} \to K, \text{ with } \Phi_K(\hat{x}) = p_0 + F_K\hat{x},$$

where F_K is the Jacobian matrix of Φ_K. The matrix F_K is given by

$$F_K := (p_1 - p_0 \vdots p_2 - p_0 \vdots p_3 - p_0) \in \mathbb{R}^{3\times 3}.$$

If $\hat{x}$ is a coordinate on the reference element $\hat{K}$, then $x = \Phi_K(\hat{x})$ is the corresponding coordinate on the physical element K.

Remark. Using affine transformations simplifies the analysis of the finite element method. One reason is, that the Jacobian and the determinant are constant over the whole reference element. Furthermore, polynomials in $\hat{K}$ are mapped to polynomials of the same degree in K.

For proofing how gradient fields are transformed, we need to know how the inverse transformations look like. The inverses are given by

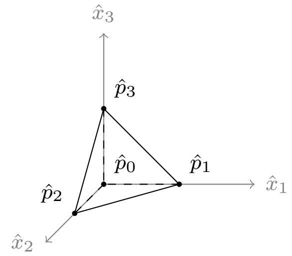
$$\Phi_K^{-1} : K \to \hat{K}, \text{ with } \Phi_K^{-1}(x) = F_K^{-1}(x - p_0).$$

3.3.1 Barycentric Coordinates

It is useful to replace Euclidean coordinates by *barycentric coordinates*, because we know that they are invariant under affine transformations – cf. [1]. That means that the coordinates of the point x are equal to the coordinates of $\hat{x}$. Our reference simplex $\hat{K}$ resp. the convex hull of vertices is defined as the set

$$\hat{K} := \{\hat{x} \mid 0 \le \hat{x}_i \le 1, \, 0 \le 1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3 \le 1\}.$$

We get the barycentric coordinates by solving the linear system



$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\lambda}_0 \\ \hat{\lambda}_1 \\ \hat{\lambda}_2 \\ \hat{\lambda}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix},$$

where $\hat{\lambda}_i$ is the barycentric coordinate with respect to the vertex V_i. $\hat{\lambda}_i(\hat{x})$ is defined as the unique linear polynomial $\hat{\lambda}_i \in \mathbb{P}_1(\hat{K})$ such that

$$\hat{\lambda}_i(V_j) = \delta_{ij} \quad \forall\, 1 \leq i, j \leq 4.$$

As a consequence of the system of equations is that

$$\sum_{i=1}^{4} \hat{\lambda}_i(\hat{x}) = 1 \quad \forall \hat{x} \in \hat{K}.$$

The barycentric coordinates are unique if and only if the four vertices are independent (ger.: *allgemeine Lage*). Our points are independent, hence the barycentric coordinates are unique. Since $\hat{\lambda}_i$ is a polynomial, we can extend it to a global function on our computational domain ω as follows:

$$\hat{\lambda}_i \in \mathcal{C}(\Omega) \quad \text{with} \quad \operatorname{supp}(\hat{\lambda}_i) = \bigcup_{\hat{K}: V_i \in \hat{K}} \hat{K} \quad \text{and} \quad \hat{\lambda}_i(V_j) = \delta_{ij} \quad \forall\, V_j \in \mathcal{V},$$

where $\mathcal{V}$ denotes the set of vertices.
In our regime we have the barycentric coordinates

$$\hat{\lambda}_1 = 1 - \hat{x}_1 - \hat{x}_2 - \hat{x}_3, \quad \hat{\lambda}_2 = \hat{x}_1, \quad \hat{\lambda}_3 = \hat{x}_2, \quad \hat{\lambda}_4 = \hat{x}_3.$$

3.3.2 Tangential and Normal Vector

For our local spaces it is important to know, how the unit tangential vector and how the outer unit normal vector are transformed – cf. global functionals. Because of the importance of these two vectors, we will look how the corresponding physical vectors are calculated.

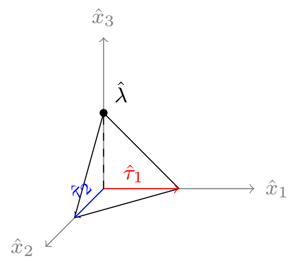
Let $\hat{\tau}$ be a unit tangential vector on the reference element and n be a unit normal vector on the reference element. We get the corresponding unit vectors on the physical elements by

$$\tau = \frac{F_K \hat{\tau}}{||F_K \hat{\tau}||}, \qquad n = \frac{F_K^{-T} \hat{n}}{||F_K^{-T} \hat{n}||}.$$

To verify the transformation of tangential vectors, one has to show for every tangential vector $\hat{\tau}$, that the corresponding $\tau = F_K \hat{\tau}$ is also a tangential vector.
If we consider the reference tetrahedron $\hat{K}$ in Figure 3.1, the tangential vector

$\hat{\tau} := \hat{x}_1 - \hat{x}_0 = (1,0,0)^T$ is mapped on $F_K(1,0,0)^T = x_1 - x_0$. Hence, $\tau = x_1 - x_0$ is also a tangential vector. For the remaining tangential vectors one can use the same arguments.

The transformation of normal vectors can not be seen so easy. Here we only want to motivate that normal vectors are transformed like gradients. The proof of this transformation can be seen in Subsection 3.3.3. Let us consider the following tetrahedron:



Because of the definition of $\hat{\lambda}$ we have

$$\partial_{\tau_1}\hat{\lambda} = 0 \quad \text{and} \quad \partial_{\tau_2}\hat{\lambda} = 0.$$

We know that the tangential derivative can be written as

$$\nabla\hat{\lambda}\cdot\tau_1 = 0 \quad \text{and} \quad \nabla\hat{\lambda}\cdot\tau_2 = 0.$$

From linear algebra we know that two vectors are orthogonal, if and only if the scalar product of these two vectors is zero. Hence

$$\nabla\hat{\lambda} \perp \tau_1 \quad \text{and} \quad \nabla\hat{\lambda} \perp \tau_2.$$

Summarizing, $\hat{n} := \nabla\hat{\lambda}$ is the normal vector of the face spanned up by τ_1 and τ_2. Thus, the vector $\hat{n}$ has to be transformed like a gradient.

3.3.3 $H(curl)$ conforming Transformation

In H^1, the transformation is simply the change of variables, i.e.

$$u(x) := \hat{u}(\Phi_K^{-1}(x)).$$

Lemma 3.7. *By using the transformation*

$$u(x) = \hat{u}(\Phi_K^{-1}(x)),$$

gradients are transformed like

$$\nabla u = F_K^{-T}\,\hat{\nabla}\hat{u}(\Phi_K^{-1}(x)).$$

Proof. Remember that $\Phi_K^{-1} := F_K^{-1}(x - x_0)$. We consider the l–th component of ∂_{x_k} from $\Phi_K^{-1}(x)$, i.e.

$$\partial_{x_k}\left(\Phi_K^{-1}(x)\right)_l = \partial_{x_k}\left(\sum_{j=1}^{3} F_K^{-1}[l,j](x - x_0)_j\right) = F_K^{-1}[l,k],$$

where $l, k = 1, 2, 3$. Now we consider $\partial_{x_k} u(x)$. Using the chain rule we get

$$\begin{aligned}
\partial_{x_k} u(x) = \partial_{x_k}\hat{u}(\Phi_K^{-1}(x)) &= \partial_{x_k}\hat{u}(\hat{\xi}_1(x), \hat{\xi}_2(x), \hat{\xi}_3(x)) \\
&= \sum_{i=1}^{3} \frac{\partial \hat{u}}{\partial \hat{\xi}_i}(\hat{\xi}_1(x), \hat{\xi}_2(x), \hat{\xi}_3(x)) \frac{\partial (\Phi_K^{-1})_i}{\partial x_k} \\
&= \sum_{i=1}^{3} \frac{\partial \hat{u}}{\partial \hat{\xi}_i}(\hat{\xi}_1(x), \hat{\xi}_2(x), \hat{\xi}_3(x)) F_K^{-1}[i,k] \\
&= \left(F_K^{-T}\hat{\nabla}\hat{u}\right)_k .
\end{aligned}$$

□

Since an element of $H(\mathrm{curl}, \Omega)$ can be written as ∇w for a $w \in H^1(\Omega)$, we suggest for $H(\mathrm{curl})$ the transformation

$$u(x) := F_K^{-T}\,\hat{u}(\Phi_K^{-1}(x)). \tag{3.4}$$

A proof of the following lemma can be found in [3].

Lemma 3.8. *By using the transformation* (3.4)*, the curl is transformed in the way of the* Piola transformation*, i.e.*

$$\mathit{curl}\, u(x) = (\det F_K)^{-1} F_K\, \hat{\mathit{curl}}\, \hat{u}(\Phi_K^{-1}(x)).$$

Knowing how elements in $H(\mathrm{curl}, \hat{K})$ and the corresponding curl are transformed we show an important property of our transformation.

Theorem 3.9. *Let $\hat{K}$ be a reference element and K be a physical element. If an $\hat{u} \in H(curl, \hat{K})$ is transformed in the way of* (3.4)*, then $u \in H(curl, K)$.*

Proof. For $\varphi \in \left[\mathcal{C}^\infty(\hat{K})\right]^3$ we have

$$\begin{aligned}
\int_K u(x) \cdot \mathrm{curl}\,\varphi(x)\,\mathrm{d}x &= \int_{\Phi_K^{-1}(K)} F_K^{-T}\hat{u}(\hat{x}) \cdot (\det F_K)^{-1} F_K \hat{\mathrm{curl}}\,\hat{\varphi}(\hat{x}) |\det F_K^{-1}|\,\mathrm{d}\hat{x} \\
&= \frac{|\det F_K^{-1}|}{\det F_K} \int_{\Phi_K^{-1}(K)} \hat{u}(\hat{x}) \cdot \hat{\mathrm{curl}}\,\hat{\varphi}(\hat{x})\,\mathrm{d}\hat{x} \\
&= \frac{|\det F_K^{-1}|}{\det F_K} \int_{\Phi_K^{-1}(K)} \hat{\mathrm{curl}}\,\hat{u}(\hat{x}) \cdot \hat{\varphi}(\hat{x})\,\mathrm{d}\hat{x}
\end{aligned}$$

$$= \frac{|\det F_K^{-1}|}{\det F_K} \int_K (\det F_K)\, F_K^{-1} \mathrm{curl}\, u(x) \cdot F_K^T \varphi(x) |\det F_K|\, \mathrm{d}x$$
$$= |\det F_K|^{-1} |\det F_K| \int_K \mathrm{curl}\, u(x) \cdot \varphi(x)\, \mathrm{d}x$$

Thus

$$\int_K u(x) \cdot \mathrm{curl}\, \varphi(x)\, \mathrm{d}x = \int_K \mathrm{curl}\, u(x) \cdot \varphi(x)\, \mathrm{d}x,$$

and $u \in H(\mathrm{curl}, K)$. □

To close the transformation section we want to show that this transformation is $H(\mathrm{curl})$ conform, i.e.

- Gradient fields on $\hat{K}$ are mapped onto gradient fields on K

  Since we defined the transformation exactly the same as the transformation of gradients, this is satisfied.

- The degrees of freedom are preserved by the transformations

  Using our transformations, we see that tangential traces along edges transform as

$$(u \cdot \tau)(\Phi_K(\hat{x}))\Big|_{\Phi_K(\hat{E})} = \left(F_K^{-T} \hat{u}(\hat{x}) \cdot \frac{F_K \hat{\tau}}{||F_K \hat{\tau}||} \right) \Big|_{\hat{E}}$$
$$= \left(\hat{u}(\hat{x}) \cdot \hat{\tau} \frac{1}{||F_K \hat{\tau}||} \right) \Big|_{\hat{E}}.$$

  With our setting $F_K = I$, therefore $|\det F_K^{-1}| = 1$. Since $F_K = I$ and $\hat{\tau}$ is a unit normal vector, we have that $||F_K \hat{\tau}|| = ||\hat{\tau}|| = 1$. Thus, $|\det F_K^{-1}| = ||F_K \hat{\tau}||$.

  If we consider the degrees of freedom, we have

$$\int_E u \cdot \tau\, \mathrm{d}\ell_x = \int_{\Phi_K^{-1}(E)} \hat{u}(\hat{x}) \cdot \hat{\tau} \frac{1}{||F_K \hat{\tau}||} ||F_K \hat{\tau}||\, \mathrm{d}\ell_{\hat{x}} = \int_{\hat{E}} \hat{u}(\hat{x}) \cdot \hat{\tau}\, \mathrm{d}\ell_{\hat{x}}.$$

3.4 Interpolation Error Estimates

With our setting for the finite element method, it would be possible to show Theorem 5.41 in [3]. Here we only consider a special case of this theorem, for the general one and the proof see [3].

Theorem 3.10. *Let $\mathcal{T}_h$ be an admissible mesh on Ω and let $u \in [\mathcal{C}^1(\overline{\Omega})]^3$. Then*

$$||u - I^{curl} u||_{[L^2(\Omega)]^3} + ||curl(u - I^{curl} u)||_{[L^2(\Omega)]^3}$$
$$\leq C \cdot h \left(||u||_{[H^1(\Omega)]^3} + ||curl\, u||_{[H^1(\Omega)]^3} \right)$$

Chapter 4

Conclusion

In this bachelor thesis we introduced physical laws which describe the phenomena of electromagnetism. Combining these laws led to the Maxwell's equations in their classical form. A reformulation of these equations and using the material laws brought us to the de Rham complex and the vector potential formulation. Many different types of problems have the same structure. Thus, it was in our interest to analyze the curl–curl problem.

Looking at the curl–curl problem we realized that we need a different integration by parts formula. We also introduced a trace operator for functions in our function space $H(\text{curl}, \Omega)$ to have well defined expressions in our weak formulation. In view of the finite element method we looked for an interface condition, such that our computational domain can be decomposed in several smaller domains without loosing conformity.

When speaking about finite elements for Maxwell's equations, we needed to define local element spaces and global functionals to derive the finite element spaces. A main goal was to achieve an exact discrete sequence – like the exact de Rham sequence in the infinite dimensional case. Therefore we introduced nodal interpolation operators which should satisfy a commuting property. The global functionals were derived by starting in the discrete space of $L_2(\Omega)$ using the famous integral theorems of Stokes and Gauss. An important step was to prove, if the finite element space is a subset of the corresponding Sobolev space. For implementation aspects we considered transformations from a reference element on the physical elements and showed, that these transformations preserve the degrees of freedoms. Finally, we cited an interpolation error estimate.

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Eidesstattliche Erklärung

Ich, Florian Kagerer, erkläre an Eides statt, dass ich die vorliegende Bachelorarbeit selbständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Asten, Juli 2018

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